Perfectoid Spaces

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$\boxed{HW10}$: $|\cdot| : A \to \Gamma$ valuation $\Rightarrow c\Gamma = \{\gamma \in \Gamma \mid \exists_{m,n \in \mathbb{Z}}$ with $m < 0 < n, \exists a \in A : |a| > 1$ and $|a|^m \leq \gamma \leq |a|^n\}$

$\boxed{HW11}$: $|\cdot| : A \to \Gamma$ valuation, $H \leq \Gamma$ convex $\Rightarrow$ TFAE

(i) $|\cdot|_H$ is a valuation

(ii) $H \supseteq c\Gamma$

(iii) for $|\cdot|^* : A \xrightarrow{|\cdot|} \Gamma \to \Gamma/H$ we have $|A|^* \leq 1$

(iv) $|a| \notin H \Rightarrow |a| < 1$

$\boxed{\text{Prop}}$: A f-adic, $\Sigma \subseteq A$, A_Σ = smallest open subring of A that is integrally closed in A

$\Rightarrow A_\Sigma = \{a \in A \mid \forall |\cdot| \in \mathrm{Spa}(A,\Sigma) : |a| \leq 1\}$

$\boxed{\text{Cor}}$: (A, A^+) affinoid, $f_1, \dots, f_n, g \in A$ s.t. $(f_1, \dots, f_n) \underset{\text{open}}{\subseteq} A$. [Recall that $A\left(\frac{f_1,\dots,f_n}{g}\right)^+ :=$ integral closure of $A^+\left[\frac{f_i}{g} \mid i = 1, \dots, n\right]$ in $A\left(\frac{f_1,\dots,f_n}{g}\right)$ $\left(\subseteq A\left(\frac{f_1,\dots,f_n}{g}\right)^\circ \text{ \& open}\right)$

By prop above, $A\left(\frac{f_1,\dots,f_n}{g}\right)^+ = \left\{a \in A\left(\frac{f_1,\dots,f_n}{g}\right) \mid \forall |\cdot| \in \mathrm{Spa}\left(A\left(\frac{f_1,\dots,f_n}{g}\right), A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]\right) : |a| \leq 1\right\}$]

Consider $\varphi : (A, A^+) \to \left(\overbrace{A\left(\frac{f_1,\dots,f_n}{g}\right)}^{B=}, \overbrace{A\left(\frac{f_1,\dots,f_n}{g}\right)^+}^{B^+=}\right)$ the natural adic morphism of affinoid rings. Then: $\mathrm{Spa}(\varphi)$ induces $\mathrm{Spa}\left(A\left(\frac{f_1,\dots,f_n}{g}\right), A\left(\frac{f_1,\dots,f_n}{g}\right)^+\right) \xrightarrow{\text{homeo}} R\left(\frac{f_1,\dots,f_n}{g}\right) =: U \subseteq \mathrm{Spa}(A,A^+)$ and $\mathrm{Spa}(\varphi)$ preserves rational open sets.

$\boxed{\text{Lemma}}$: $(A, A^+) \xrightarrow{\varphi} (B, B^+)$ an adic morphism of affinoid rings (i.e. $\varphi(A^+) \subseteq B^+$ & $\underbrace{\varphi(\text{ideal of def}) \text{ is open}}$)

this is the 'adic' part about φ

$\Rightarrow \mathrm{Spa}(\varphi)^{-1}\left(\begin{smallmatrix}\text{rational}\\ \text{open}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\text{rational}\\ \text{open}\end{smallmatrix}\right)$

$\boxed{\text{Pf}}$: $f_1, \dots, f_n, g \in A$ s.t. $(f_1, \dots, f_n) \subseteq A$ open. φ adic $\Rightarrow (\varphi(f_1), \dots, \varphi(f_n)) = \varphi(f_1, \dots, f_n) \cdot B$ open.

We want to show that $R\left(\frac{\varphi(f_1), \dots, \varphi(f_n)}{\varphi(g)}\right)^{(1)} = \mathrm{Spa}(\varphi)^{-1}\left(R\left(\frac{f_1,\dots,f_n}{g}\right)\right)^{(2)}$, which is obvious. □

$\boxed{\text{Pf}}$ (of Cor): Let us start by showing that $\mathrm{Spa}(\varphi)$ is injective. Note that $B = A[g^{-1}]$ as rings.

Take $|\cdot| \in \mathrm{Spa}(B, B^+)$, and denote $|\cdot|^* = \mathrm{Spa}(\varphi)(|\cdot|) = |\cdot| \circ \varphi$. Then $|\cdot|^*$ determines $|\cdot|$ because $\left|\frac{a}{g^n}\right| = \frac{|a|^*}{(|g|^*)^n}$ $\rightsquigarrow$ injectivity ✓

Now let us prove $\mathrm{Im}\,\mathrm{Spa}(\varphi) \subseteq R\left(\frac{f_1,\dots,f_n}{g}\right)$: take $|\cdot| \in \mathrm{Spa}(B, B^+)$. As $g \in B^\times \Rightarrow |g| \neq 0$.

$\Rightarrow$ enough: $|f_i| \leq |g|$, or equivalently $\left|\frac{f_i}{g}\right| \leq 1$. Fine as $\frac{f_i}{g} \in B^+$. $\rightsquigarrow \subseteq R\left(\frac{f_1,\dots,f_n}{g}\right)$ ✓

Now we turn to $\mathrm{Spa}(\varphi)$ surjects onto $R\left(\frac{f_1,\dots,f_n}{g}\right)$: Take $|\cdot| \in R\left(\frac{f_1,\dots,f_n}{g}\right)$.

↳ Need: $|\cdot|$ extends to $A\left(\frac{f_1,\dots,f_n}{g}\right)$ continuously & $\left|A\left(\frac{f_1,\dots,f_n}{g}\right)^+\right| \leq 1$.

As $B = A[g^{-1}]$ & $|g| \neq 0$ $\Rightarrow$ $|\cdot|$ extends as a valuation to B with the formula $\left|\frac{a}{g^n}\right|^* = \frac{|a|}{|g|^n}$.

Furthermore, $|B^+|^* \leq 1$ by Prop. So we are left to show continuity of the extension:

By continuity of $|\cdot|$, we have that $V = \{a \in A \mid |a| < \gamma\}$ open in A ~~for~~ $\forall \gamma \in \Gamma$.

By def of the topology of B, $W = V \cdot$ "subring of B gen'd by $\frac{f_i}{g}$" is open in B.

But then $|W|^* < \gamma$, so $|\cdot|^*$ is continuous $\rightsquigarrow$ ~~subset~~ $\mathrm{Spa}(\varphi)$ surjective onto $R\left(\frac{f_1, \dots, f_n}{g}\right)$.

We are left to show: $\mathrm{Spa}(\varphi)\left(\begin{smallmatrix}\text{rational}\\ \text{open}\end{smallmatrix}\right) = \begin{smallmatrix}\text{rational}\\ \text{open}\end{smallmatrix}$

Take $R\left(\frac{t_1, \dots, t_N}{s}\right) \in B$. As $g \in B^\times$, we may multiply $t_1, \dots, t_N, s$ by any power of g

↳ wma $t_i \in A$, $s \in A$. As $(t_1, \dots, t_n)$ is open, it contains a subset of the form

$(t_1, \dots, t_n) \supseteq V$ "subring generated by $\frac{t_1}{s}, \dots, \frac{t_N}{s}$". $\forall a \in V\ \exists n, m \geq 0$ s.t. $a \cdot g^n = g^n \sum^{m} \overset{\in A}{a_i}\, t_i$.

$\boxed{\text{Lemma}}$: (A, A^+) affinoid ~~...~~, ~~...~~ $X \underset{\text{open}}{\subseteq} \mathrm{Spa}(A, A^+)$ s.t. X quasi-compact, $s \in A$ s.t. $|s| \neq 0$ $\forall |\cdot| \in X$. $\Rightarrow$ $\exists\, 0 \in U$ open s.t. $\forall b \in U\ \forall |\cdot| \in X$: $|b| \leq |s|$.

Finish the proof of Cor:

Thm: $\mathrm{Spa}(A, A^+)$ is quasi-compact.

So let $X = \mathrm{Spa}(A, A^+) \underset{\text{Lemma}}{\Longrightarrow} U \supseteq I^j$ where $(\tilde{A}, I)$ is a couple of def.

Choose generators $(t_{n+1}, \dots, t_m)_{\tilde{A}} = I^j$ $\Rightarrow$ $(t_1, \dots, t_m) \subseteq A$ open

$\Rightarrow$ $\mathrm{Spa}(\varphi)\left(R\left(\frac{t_1, \dots, t_N}{s}\right)\right) = R\left(\frac{f_1, \dots, f_n}{g}\right) \cap R\left(\frac{t_1, \dots, t_m}{s}\right)$ $\square$ (Pf of Cor)

$\boxed{\text{Pf}}$ (of Lemma): Let $|\cdot| \in \mathrm{Spa}(A, A^+)$.

Consider $\{a \in A \mid |a| < |s|\}$ is open $\Rightarrow$ $\exists\, n_{|\cdot|}$ s.t. $I^{n_{|\cdot|}} \subseteq \{a \in A \mid |a| < |s|\}$.

Define $n_{|\cdot|} := \inf\{n \mid I^n \subseteq \{a \in A \mid |a| < |s|\}\}$. What we have to show is that

$X \ni |\cdot| \longrightarrow n_{|\cdot|} \in \mathbb{N}$ bounded from above. Define $X_n = \{|\cdot| \in X \mid n_{|\cdot|} \leq n\}$

↳ $X = \bigcup_n X_n$. So it is enough to show that $X_n \underset{\text{open}}{\subseteq} \mathrm{Spa}(A, A^+)$.